Continuous Beam Deflection Monitoring Using Precise Inclinometers.

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SUMMARY

When a beam is supported at one or more points along its length and subjected to vertical loads, it bends. This bending results in a series of vertical displacements along its length.

In many engineering situations, the precise nature of the deflections must be determined. For example, in beams supporting floors in buildings, excess deflection will cause cracking of the plaster.

Following the loading of key points in a structure, rotations and deflections of the beam due to such loads are measured. The measured movements are then compared to the movements expected based on design criteria.

Rather than focusing on the performance of a structure under specific loading conditions - which in many cases is not feasible, a continuous monitoring approach evaluates long-term behaviour and changes. For this purpose one or more precise inclinometers can be permanently installed on or within the structure.

Rotations can be used to quantify angular displacements, but it is often the case that linear displacements are the more desired results. Without a stable reference, linear movement is difficult, and even impossible, to measure with conventional sensors. By referencing gravity, precise inclinometers avoid the stable reference problem.

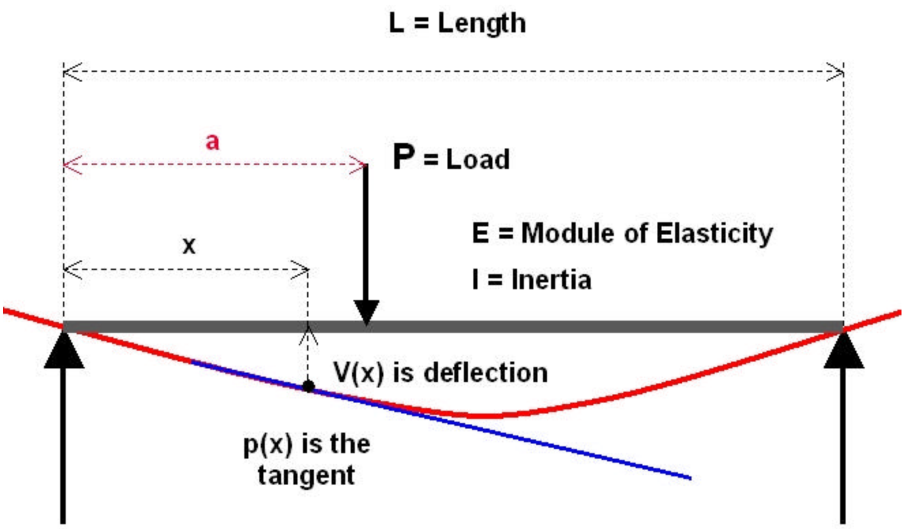
The author reviews the results of the analysis of a series of loading experiments, including simply supported beams with point loads or uniformly distributed loads. Formulas for the beam deflections, and the locations and magnitudes of the vertical shear and the maximum bending stresses, are described in terms of rotational parameters that can be measured with the new Leica NIVEL210 and NIVEL220 precise inclinometers.

Up to 30 Leica NIVEL220 precise inclinometers - each having an onboard RS-485 interface - can be networked along a single cable to provide an effective continuous monitoring solution.

1. THEORETICAL BACKGROUND

1.1 Beam Deflection

For a given beam of length l loaded by a stress P located at $x = a$ we understand by the strength of materials that the flexure of the structure $V(x)$ is composed by two segments of parabolas, having as definition domain the intervals $[0, a]$ and $]a, b]$.



These parabolas are of course are continuous at $x = a$ meaning that they have for this point the same flexure $v(a)$ and the same tangent $p(a)$.

The analytical expressions are:

For $0 \leq x \leq a$ we have:

$$v_1(x) = \frac{-P(l-a).x}{6.l.EI} \cdot \left[a(2l-a) - x^2\right] \quad (1)$$

$$p_1(x) = \frac{\partial v(x)}{\partial x} = \frac{-P}{6.l.EI} \cdot (l-a)\left[a(2l-a) - 3x^2\right] \quad (2)$$

For $a \leq x \leq l$ we have:

$$v_2(x) = \frac{-Pa(l-x)}{6.l.EI} \cdot \left[l^2 - a^2 - (l-x)^2\right] \quad (3)$$

$$p_2(x) = \frac{\partial v(x)}{\partial x} = \frac{-Pa}{6.l.EI} \cdot \left[a^2 - l^2 + 3(l-x)^2\right] \quad (4)$$

At $x = a$, we have the following unique value:

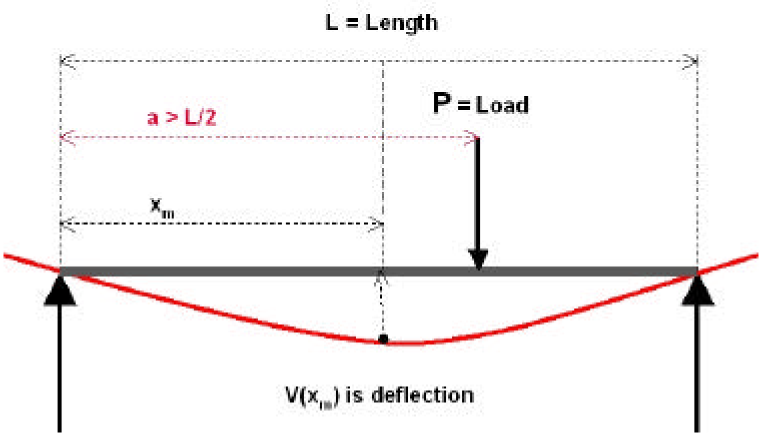
$$v(a) = \frac{-Pa^2}{3.l.EI} \cdot (l-a)^2$$

$$p(a) = \frac{-Pa}{3.l.EI} \cdot (l-a) \cdot (l-2a)$$

What will be the value of x_m where there is the maximum flexure?

There are two cases:

- If $a \geq \frac{l}{2}$, then the maximum flexure will occur for $x_m < a$



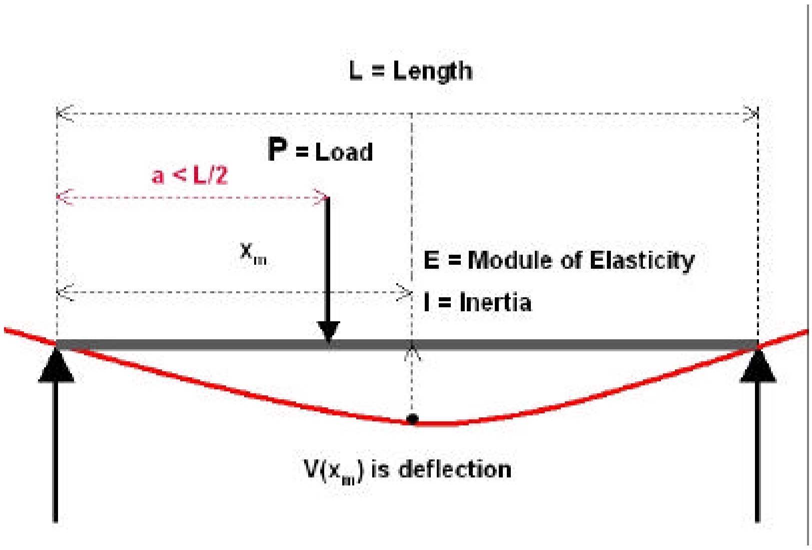
To determine the x_m value, one needs to find the point at which the tangent value will be zero $p_1(x) = 0$.

Hence $$x_m = \frac{1}{3} \cdot \sqrt{a(2l-a)} \qquad (5)$$

By substituting x_m into the expression for $V_1(x)$ we obtain for the maximum flexure:

$$v_1(x_m) = \frac{-P \cdot (l-a) \cdot [a(2l-a)]^{\frac{3}{2}}}{9 \cdot \sqrt{3} \cdot EI \cdot l} \qquad (6)$$

- If $a \leq \frac{l}{2}$, then the maximum flexure will occur for $x_m > a$



To determine the x_m value, one needs to find the point where the tangent value will be zero, $p_2(x) = 0$.

Hence $$x_m = l - \frac{\sqrt{l^2 - a^2}}{\sqrt{3}} \quad (7)$$

By substituting x_m into the expression for $V_2(x)$ we obtain for the maximum flexure:

$$v_2(x_m) = \frac{-Pa\left(l^2 - a^2\right)^{\frac{3}{2}}}{9 \cdot \sqrt{3} \cdot EI \cdot l} \quad (8)$$

1.2 Expression for an Inclinometer when $x = b < a$

Assume a beam loaded by a stress P located at the abscissa $x = a$ and an inclinometer located at $x = b$ measuring a tilt value of i_1 what will be the expression for the maximum flexure?

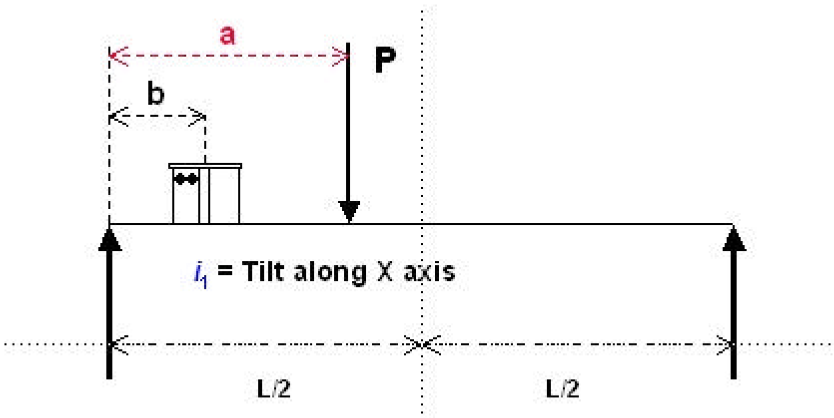
The formula of the tangent, equation (2), gives:

$$i_1 = \frac{-P}{6 \cdot EI \cdot l}(l-a)\cdot\left[a(2l-a)-3b^2\right]$$

Then we obtain:

$$\frac{-P}{EI \cdot l} = \frac{6i_1}{(l-a)\cdot\left[a(2l-a)-3b^2\right]}$$

To determine the maximum flexure, one must replace the value of $\frac{-P}{EI \cdot l}$ in the formula of the flexure (equations (6) and (8)) for the two different situations:



- If $a \geq \frac{l}{2}$

$$v_1 \max = \frac{2i_1 \cdot \left[a(2l-a)\right]^{\frac{3}{2}}}{3\sqrt{3}\cdot\left[a(2l-a)-3b^2\right]} \qquad (9)$$

- If $a \leq \frac{l}{2}$

$$v_2 \max = \frac{2i_1 \cdot a\left(l^2-a^2\right)^{\frac{3}{2}}}{3\sqrt{3}\cdot(l-a\cdot)\left[a(2l-a)-3b^2\right]} \qquad (10)$$

1.3 Expression for an Inclinometer when $x = b > a$

The development is as for section 1.2, except that in this case the inclinometer is located on the right hand side of a so that $(b > a)$.

We have (by expanding equation (4)):

$$i_2 = \frac{-P \cdot a}{6 \cdot EI \cdot l} \cdot \left[a^2 - l^2 + 3(l-b)^2\right]$$

Then we obtain:

$$\frac{-P}{EI \cdot l} = \frac{6i_2}{a \cdot \left[a^2 - l^2 + 3 \cdot (l-b)^2\right]}$$

To determine the maximum flexure, we must replace the value of $\frac{-P}{EI \cdot l}$ in the formula of the flexure (equations (6) and (8)) for the two different situations:

- If $a \geq \frac{l}{2}$

$$v_1 \max = \frac{2i_2 \cdot (l-a) \cdot \left[a(2l-a)\right]^{\frac{3}{2}}}{a \cdot 3\sqrt{3} \cdot \left[a^2 - l^2 + 3 \cdot (l-b)^2\right]} \qquad (11)$$

- If $a \leq \frac{l}{2}$

$$v_2 \max = \frac{2i_2 \cdot \left(l^2 - a^2\right)^{\frac{3}{2}}}{3\sqrt{3} \cdot \left[a^2 - l^2 + 3 \cdot (l-b)^2\right]} \qquad (12)$$

1.4 Remarks

Whenever the load P is located at a along the beam, the position x_m, where the maximum flexure will be, is close to the middle of the beam.

In fact $\lim_{a \to 0} x_m = l - \frac{\sqrt{l^2 - a^2}}{\sqrt{3}} = l - \frac{l}{\sqrt{3}} = 0.4226 \cdot l$

So the maximum distance from the middle of the beam is $(0.5 - 0.4226) \cdot l = 0.077 \cdot l$

1.5 Summary

The various cases developed, by taking into account where the load is located along the beam as well as where the inclinometer will be setup, are summarised here.

a and b are along the beam

when $x \le a$

$$v_1(x) = \frac{-P \cdot (l-a) \cdot x \cdot \left[a(2l-a) - x^2\right]}{6 \cdot EI \cdot l}$$

$$p_1(x) = \frac{-P \cdot (l-a) \cdot \left[a(2l-a) - 3x^2\right]}{6 \cdot EI \cdot l}$$

when $x \ge a$

$$v_2(x) = \frac{-P \cdot a \cdot (l-x) \cdot \left[l^2 a^2 - (l-x)^2\right]}{6 \cdot EI \cdot l}$$

$$p_2(x) = \frac{-P \cdot a \cdot \left[a - l^2 + 3(l-x)^2\right]}{6 \cdot EI \cdot l}$$

when $a \le \frac{l}{2}$

$$x_{\max} = \frac{l - \sqrt{l^2 - a^2}}{\sqrt{3}}$$

$$v(x_{\max}) = \frac{-P \cdot a \cdot \left(l^2 - a^2\right)^{\frac{3}{2}}}{9\sqrt{3} \cdot EI \cdot l}$$

when $a \ge \frac{l}{2}$

$$x_{\max} = \frac{\sqrt{a(2l-a)}}{\sqrt{3}}$$

$$v(x_{\max}) = \frac{-P \cdot (l-a) \cdot \left[a(2l-a)\right]^{\frac{3}{2}}}{9\sqrt{3} \cdot EI \cdot l}$$

when $b \le a$

$$i_1 = \frac{-P}{6 \cdot EI \cdot l} \cdot (l-a) \cdot \left[a(2l-a) - 3b^2\right]$$

$$v_{\max} = \frac{2i_1 \cdot a \cdot \left(l^2 - a^2\right)^{\frac{3}{2}}}{3\sqrt{3} \cdot (l-a) \cdot \left[a(2l-a) - 3b^2\right]}$$

$$v_{\max} = \frac{2i_1 \cdot \left[a \cdot (2l-a)\right]^{\frac{3}{2}}}{3\sqrt{3} \cdot \left[a(2l-a) - 3b^2\right]}$$

when $b \ge a$

$$i_2 = \frac{-P}{6 \cdot EI \cdot l} \cdot \left[a^2 - l^2 + 3(l-b)^2\right]$$

$$v_{\max} = \frac{2i_2 \cdot \left(l^2 - a^2\right)^{\frac{3}{2}}}{3\sqrt{3} \cdot \left[a^2 - l^2 + 3(l-b)^2\right]}$$

$$v_{\max} = \frac{2i_2 \cdot (l-a) \cdot \left[a(2l-a)\right]^{\frac{3}{2}}}{3\sqrt{3} \cdot a \cdot \left[a^2 - l^2 + 3(l-b)^2\right]}$$

for $a = \frac{l}{2}$ **and** b **of any value**

when $x \le a = \frac{l}{2}$

$$v_1(x) = \frac{-P \cdot x \cdot \left[3l^2 - 4x^2\right]}{48 \cdot EI}$$

$$p_1(x) = \frac{-P \cdot \left(l^2 - 4x^2\right)}{16 \cdot EI}$$

when $x \ge a = \frac{l}{2}$

$$v_2(x) = \frac{-P \cdot (l-x) \cdot \left[3l^2 - 4(l-x)^2\right]}{48 \cdot EI}$$

$$p_2(x) = \frac{-P \cdot \left[4(l-x)^2 - l^2\right]}{16 \cdot EI}$$

when $a = \frac{l}{2}$

$$x_m = \frac{l}{2} \quad \text{and} \quad v_{x_{\max}} = \frac{-P \cdot l^3}{48 \cdot EI}$$

when $b \le a$

$$i_1 = \frac{-P \cdot \left(l^2 - 4b^2\right)}{16 \cdot EI}$$

$$v_{\max} = \frac{i_1 \cdot l^3}{3\left(l^2 - 4b^2\right)}$$

when $b \ge a$

$$i_2 = \frac{-P \cdot \left[4(l-b)^2 - l^2\right]}{16 \cdot EI}$$

$$v_{\max} = \frac{i_2 \cdot l^3}{3\left[4(l-b)^2 - l^2\right]}$$

2. EXAMPLES

2.1 Case 1

On a beam HEB 300 of 8 meters length, with $I = 25166 \cdot 10^{-8} m^4$ and $E = 21 \cdot 10^6 \, T/m^2$ a load is applied at the midpoint $\left(a = l/2\right)$. We has installed an inclinometer at $b = 0$ that measures a tilt angle $i_1 = -0.0095$.

<u>Question 1:</u> What will be the maximum flexure?

$$V_{\max} = \frac{i_1 \cdot l^3}{3 \cdot \left(l^2 - 4b^2\right)} = \frac{-0{,}0095 \cdot 8}{3} = -0{,}0253m$$

<u>Question 2:</u> What will be the load?

$$i_1 = \frac{-P \cdot \left(l^2 - 4b^2\right)}{16 \cdot EI}$$

We can determine this load by using the following formula, because we know also the characteristics of the beam:

$$EI = 25166 \cdot 10^{-8} \cdot 21 \cdot 10^6 = 5284{,}86Tm^2$$

$$P = \frac{-i_1 \cdot 16 \cdot 5284{,}86}{8^2} = \frac{-\left(-0.0095\right) \cdot 16 \cdot 5284{,}86}{8^2} = 12{,}55T$$

2.2 Case 2

We consider a structure that we don't know the rigidity of.

To determining it we need to load it only one time and measure the flexure. In that case we consider a load of 15 T applied at the center of a beam of 10 meters length, and a maximum measured flexure of –5 cm.

To determine the rigidity, we apply:

$$V_{\max} = \frac{-P \cdot l^3}{48 \cdot EI} \quad \text{then} \quad EI = \frac{-15 \cdot 10^3}{48 \cdot (-0.05)} = 6250 Tm^2$$

Supposing now that the reading of an inclinometer located at 1 meter from the support (b = 1m) gives a value of 0.012 radians (i1) and that the load is located 4 meters (a = 4 meters) from the support, we obtain:

$$x_m = l - \frac{\sqrt{l^2 - a^2}}{\sqrt{3}} = 10 - \frac{\sqrt{10^2 - 4^2}}{\sqrt{3}} = 4,7085m$$

$$V_{\max} = \frac{2 \cdot i_1 \cdot a(l^2 - a^2)^{\frac{1}{2}}}{3\sqrt{3} \cdot (l-a) \cdot [a \cdot (2l - a) - 3b^2]} = -0,0389m$$

The value of P will be:

$$i_1 = \frac{-P}{6 \cdot EI \cdot l} \cdot (l-a) \cdot [a \cdot (2l-a) - 3b^2]$$

$$P = \frac{6 \cdot 6250 \cdot 10 \cdot 0,012}{(10-4) \cdot [4(2 \cdot 10 - 4) - 3 \cdot 1^2]} = 12,29T$$

3. THE NEW LEICA PRECISE INCLINOMETER NIVEL200 SERIES

The new precise inclinometer Leica Nivel200 has been developed especially for structural monitoring applications and deformation measurements.

The new Leica Precise InclinometerNIVEL200 Series

The sensor provides simultaneous information about the inclination and its direction with reference to two axes, as well as the temperature at a point on a structure. Specially designed for use on bridges, dams, tunnels, as well as tall buildings and other large-scale structures, it provides precise information about changes in inclination, which can then be further processed using the Leica GeoMoS monitoring software.

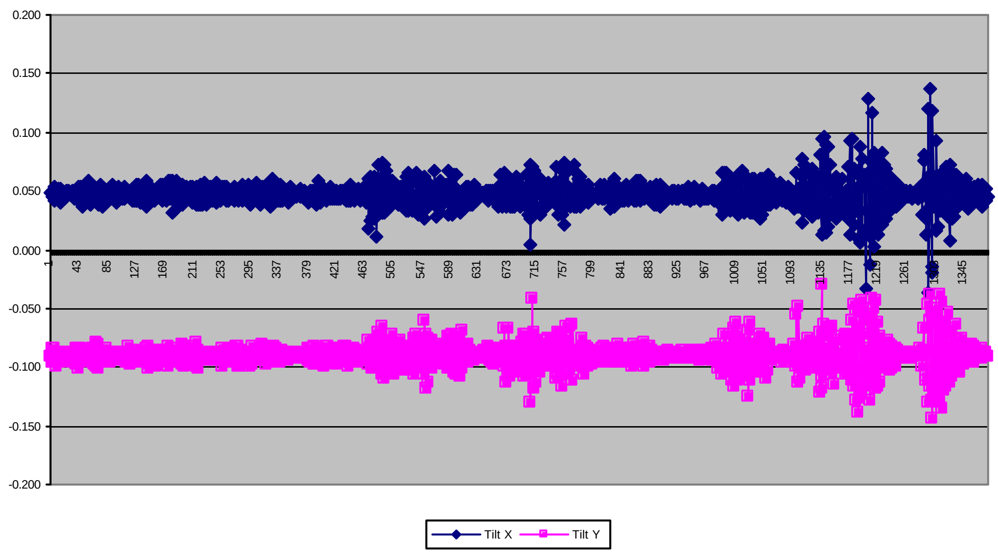
Installation of NIVEL220 in the Burj Dubai - world tallest building - by Doug Hayes

The two-axis sensor has an accuracy of 0.001 mrad and can detect a minimum vertical movement over a measuring range of +/-3 mrad.

Time	Tilt X mRad	Tilt Y mRad	T° Celsius
26/08/2006 12:00	0.048	-0.091	35.5
26/08/2006 12:00	0.048	-0.085	35.5
26/08/2006 12:00	0.050	-0.096	35.5
26/08/2006 12:00	0.045	-0.093	35.5
26/08/2006 12:00	0.042	-0.085	35.5
26/08/2006 12:00	0.054	-0.097	35.5
26/08/2006 12:00	0.044	-0.090	35.5
26/08/2006 12:00	0.051	-0.087	35.5
26/08/2006 12:00	0.046	-0.099	35.5
26/08/2006 12:00	0.048	-0.089	35.5
26/08/2006 12:00	0.046	-0.093	35.5
26/08/2006 12:00	0.052	-0.096	35.5
26/08/2006 12:00	0.048	-0.095	35.5
26/08/2006 12:00	0.048	-0.089	35.5

Example of Leica NIVEL200 measurements

The temperature reading is an important parameter for every monitoring application in order to take into account external influences on a structure. The Leica GeoMoS monitoring software continuously processes the data. As the sensors have standard interfaces they can also be easily integrated into other systems.



Example of NIVEL200 measurement time series

The two Leica Nivel200 models differ only in their connectivity options. The Leica Nivel210 has an RS232 interface, which can be connected directly to a computer, whilst the Leica Nivel220 has an RS485 interface, which can be connected to a bus system. With only one line of communication, up to 32 NIVEL220 can be supported, allowing the user to establish a complete network.

In order to make the setting up and integration of the inclination measuring system as easy as possible, Leica Geosystems has a range of suitable accessories for the Leica Nivel200, such as a mains adapter, cable and wall bracket.

The NIVEL200 has; however, a limited range of operation compared to other inclinometers but has dual axis sensors integrated and due to its unique design is the most accurate sensor available.

To cope with this limited measurement range, the development of a fundamental formula provides a means of optimizing the position of the NIVEL200 on a structure in order to maximize its performance.

This optimized position $b_{optimal}$ is obtained by optimizing the following function:

$$\frac{1}{2}\sqrt{\frac{16 \cdot EI \cdot i_{\max}}{P} + l^2} \leq b_{optimal} < \frac{1}{2}\sqrt{\frac{16 \cdot EI \cdot i_{\min}}{P} + l^2}$$

Where we suppose that the load P is applied to the middle of the beam $\left(\frac{l}{2}\right)$ and the inclinometer installed in the interval $\left]0, \frac{l}{2}\right]$.

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REFERENCE

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